Investigation 4

Factorizations: Searching for Factor Strings

Some numbers can be written as the product of several different pairs of factors. For example, 100 can be written as 1×100, 2×50, 4×25, 5×20, and 10×10. It is also possible to write 100 as the product of three factors, such as $2 \times 2 \times 25$ and $2 \times 5 \times 10$.

Getting Ready for Problem 4.1

Can you find a longer string of factors with a product of 100?

4.1 The Product Puzzle

The Product Puzzle is a number-search puzzle in which you look for strings of factors with a product of 840. Two factor strings have been marked in the puzzle at the right.

How many factor strings can you find?

The Product Puzzle

5	42	14	15	56	3
20	3	4	420	28	5
70	12	35	210	2	168
120	24	14	2	28	84
7	280	3	4	6	10
3	2	105	140	4	5
20	40	8	21	2	7

Problem 4.1 Finding Factor Strings

In the Product Puzzle, find as many factor strings for 840 as you can. When you find a string, draw a line through it. Keep a list of the strings you find.

A. What is the longest factor string you found?

B. If possible, name a factor string with a product of 840 that is longer than any string you found in the puzzle. Do not consider strings that contain 1.

C. Choose a factor string with two factors. How can you use this string to find a factor string with three factors?

D. How do you know when you have found the longest possible string of factors for a number?

E. How many distinct longest strings of factors are there for 840? Strings are *distinct* if they are different in some way other than the order in which the factors are listed.

The Product Puzzle

5	42	14	15	56	3
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ACE Homework starts on page 56.

4.2 Finding the Longest Factor String

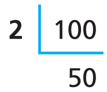
The strings of factors of a number are called **factorizations** of that number. In Problem 4.1, you saw that the longest possible factor string for 840 is made up of prime numbers. We call this string the **prime factorization** of 840. In fact, the longest factor string for any whole number is the prime factorization. Can you explain why?

Getting Ready for Problem 4.2

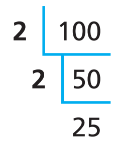
When you look for the prime factorization of a number, it helps to have a list of prime numbers handy. Look back at the table of first moves you made for the Factor Game. Make a list of all prime numbers less than 30.

One method for finding the prime factorization of a number is described below. In this example, you'll find the prime factorization of 100.

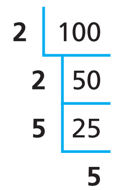
- First, find one prime factor of 100. You can start with 2. Divide 100 by 2, showing the work as an upside-down division problem.



- Next, find a prime factor of 50. You can use 2 again. Add another step to the division problem.



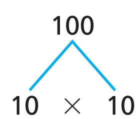
- Now, find a prime factor of 25. The only possibility is 5.



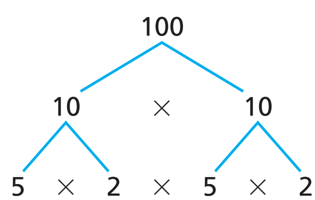
You are left with a prime number, 5. From the final diagram, you can read the prime factorization of 100: $100 = 2 \times 2 \times 5 \times 5$.

You could also use a *factor tree* to find the prime factorization of a number. Here are the steps to make a factor tree for 100.

- First, find a factor pair of 100. You might use 10 and 10. Write 100 and then draw branches from 100 to each factor.

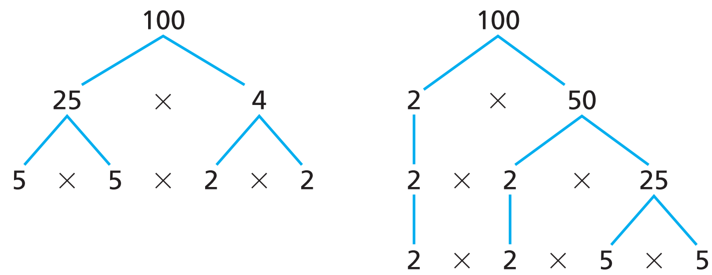


- If possible, break each factor you chose into the product of two factors. Draw branches to show how the factors are related to the numbers in the row above.



- Because all the numbers in the bottom row are prime, the tree is complete. The prime factorization of 100 is $5 \times 2 \times 5 \times 2$.

Here are two more factor trees for 100:



- In the tree at the right, notice that the 2 in the second row does not break down further. Draw a single branch, and repeat the 2 in the next rows.

You can see that the bottom row of each tree contains the same factors, although the order of the factors is different. You can also see that factor trees give you the same prime factorization for 100 as the previous method.

You can use a shorthand notation to write prime factorizations. For example, you can write $5 \times 5 \times 2 \times 2$ as $5^2 \times 2^2$. The small raised number is an exponent. An **exponent** tells you how many times a factor is used. For example, $2^2 \times 5^4$ means a 2 is used twice as a factor and a 5 is used four times. So, $2^2 \times 5^4$ is the same as $2 \times 2 \times 5 \times 5 \times 5 \times 5$.

You can read some exponents in more than one way.

Example	Ways to read
3^2	3 to the second power OR 3 squared
5^3	5 to the third power OR 5 cubed
2^4	2 to the fourth power

Problem 4.2 Finding the Longest Factor String

A. Why do we say *the* prime factorization of 100 instead of *a* prime factorization of 100?

B. Find the prime factorizations of 72, 120, and 600.

C. Write the prime factorizations of 72, 120, and 600 using exponents.

D. Choose a composite factor of 72.

1. Show how this composite factor can be found in the prime factorization of 72.
2. This composite factor is part of a factor pair for 72. How can you use the prime factorization to find the other factor in the pair?

E. Find a multiple of 72. What will the prime factorization of this multiple have in common with the prime factorization of 72?

ACE **Homework starts on page 56.**

4.3 Using Prime Factorizations

Derrick wanted to find the common factors and common multiples of 24 and 60. He made Venn diagrams similar to the ones you made in Problem 2.3. He conjectured that he could use prime factorization to find common factors.

First, he found the prime factorizations of 24 and 60.

$$24 = 2 \times 2 \times 2 \times 3$$
$$60 = 2 \times 2 \times 3 \times 5$$

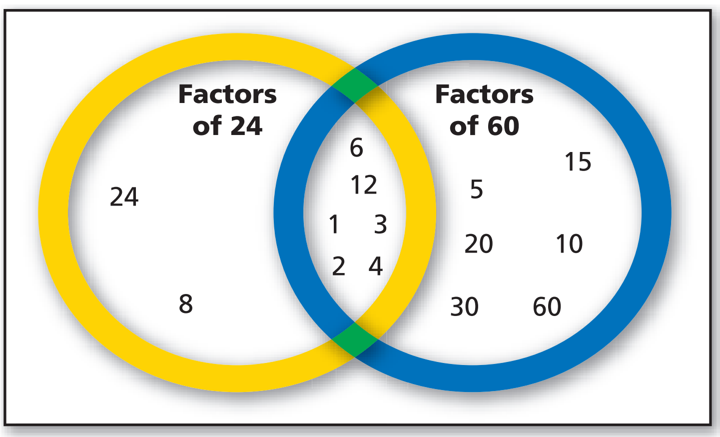
Both prime factorizations contain 2×2, which shows that 4 is a common factor.

$$24 = \boxed{2 \times 2} \times 2 \times 3$$
$$60 = \boxed{2 \times 2} \times 3 \times 5$$

Both prime factorizations contain 2×3, which shows that 6 is a common factor.

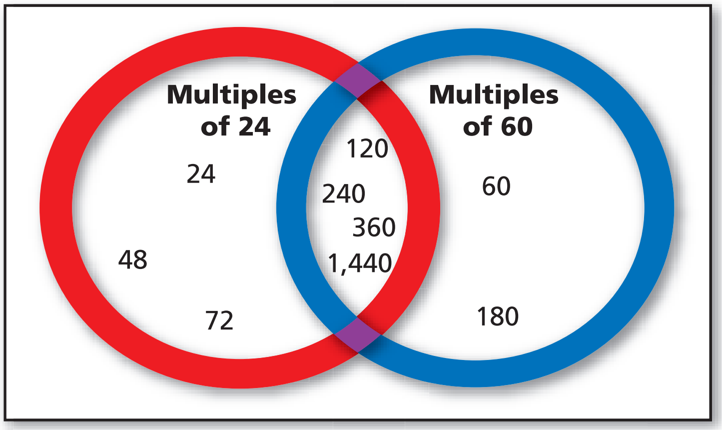
$$24 = 2 \times 2 \times \boxed{2 \times 3}$$
$$60 = 2 \times \boxed{2 \times 3} \times 5$$

Derrick noticed that the longest string common to both factorizations is $2 \times 2 \times 3$, so 12 must be the greatest common factor. He then checked a Venn Diagram and found that he was right.



Derrick wondered if he could use a similar method to find the least common multiple. He realized that the prime factorization of any multiple of 24 will include its prime factorization, $2 \times 2 \times 2 \times 3$. The prime factorization of any multiple of 60 will contain its prime factorization, $2 \times 2 \times 3 \times 5$.

So, Derrick thought the prime factorization of any common multiple should include $2 \times 2 \times 2 \times 3 \times 5$. Is he right? Check this on the Venn diagram below:



Problem 4.3 Using Prime Factorizations

A. 1. Write the prime factorizations of 72 and 120 that you found in Problem 4.2. What is the longest string common to both factorizations?

2. What is the greatest common factor of 72 and 120? How do you know?

B. 1. What is the shortest string of factors that includes the prime factorizations of both 72 and 120? Can you find a smaller common multiple of 72 and 120? Why or why not?

2. Can you find a greatest common multiple of 72 and 120? Why or why not?

C. Numbers whose greatest common factor is 1, such as 25 and 12, are **relatively prime.** How can you determine that 25 and 12 are relatively prime by looking at their prime factorizations? Find another pair of relatively prime numbers.

D. 1. Find two pairs of numbers whose least common multiple is the product of the numbers. For example, $5 \times 6 = 30$, and the least common multiple of 5 and 6 is 30.

2. Find two pairs of numbers whose least common multiple is less than the product of the numbers. For example, $6 \times 8 = 48$, but the least common multiple of 6 and 8 is 24.

3. How can you determine from the prime factorizations whether the least common multiple of two numbers is the product of the numbers or is less than the product of the two numbers? Explain your thinking.

E. If you multiply the greatest common factor of 12 and 16 by the least common multiple of 12 and 16, you get 192, which is equal to 12×16. Does this work for any two numbers? Why or why not?

ACE **Homework starts on page 56.**

Did You Know?

In all mathematics, there are a few relationships that are so basic that they are called *fundamental theorems*. There is the Fundamental Theorem of Calculus, the Fundamental Theorem of Algebra, and you have found the Fundamental Theorem of Arithmetic. The Fundamental Theorem of Arithmetic states that every whole number greater than one has exactly *one* prime factorization (except for the order in which the factors are written).

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